

Tesi

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0.1 Introduction

0.2 Freyd cover of a category

Definition 0.2.1 (Comma Category). Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ locally small categories and let $F : \mathcal{A} \rightarrow \mathcal{C}, G : \mathcal{B} \rightarrow \mathcal{C}$ two functors

$$\mathcal{A} \longrightarrow \mathcal{C} \longleftarrow \mathcal{B}$$

Then we call comma category of F and G ¹, $(F \downarrow G)$, the category with:

- Objects: an object is a triple (A, B, f) where $f : F(A) \rightarrow G(B)$ is a \mathcal{C} -morphism
- Arrows: a morphism from (A, B, f) to (A', B', f') is a pair (h, k) of an \mathcal{A} -morphism $h : A \rightarrow A'$ and a \mathcal{B} -morphism $k : B \rightarrow B'$ such that the following diagram is commutative :

$$\begin{array}{ccc} F(A) & \xrightarrow{f} & G(B) \\ F(h) \downarrow & & \downarrow G(k) \\ F(A') & \xrightarrow{f'} & G(B') \end{array}$$

Remark 0.2.2. Notice that there are two canonical forgetful functors defined over the comma category:

- a functor $\Pi_1 : (F \downarrow G) \rightarrow \mathcal{A}$ that forget the comma structure preserving the " \mathcal{A} -structure" defined as:

$$(F \downarrow G) \xrightarrow{\Pi_1} \mathcal{A}$$

$$\begin{array}{ccc} (A, B, f) & & A \\ (h, k) \downarrow & \longmapsto & \downarrow h \\ (A', B', f') & & A' \end{array}$$

¹the name "comma category" arises from the notation (F, G) used by Lawvere

- a functor $\Pi_2 : (F \downarrow G) \rightarrow \mathcal{B}$ (preserving the " \mathcal{B} -structure") defined as:

$$(F \downarrow G) \xrightarrow{\Pi_2} \mathcal{B}$$

$$\begin{array}{ccc} (A, B, f) & & B \\ \downarrow (h,k) & \xrightarrow{\quad} & \downarrow k \\ (A', B', f') & & B' \end{array}$$

Theorem 0.2.3. Let \mathcal{A} and \mathcal{B} be two finitely cocomplete categories and let $F : \mathcal{A} \rightarrow \mathcal{C}$, $G : \mathcal{B} \rightarrow \mathcal{C}$ two functors. If F is a cocontinuous functor - i.e. preserves small colimits - then the comma category $(F \downarrow G)$ is finitely cocomplete; moreover the forgetful functors are cocontinuous.

Analogously if \mathcal{A} and \mathcal{B} are finitely complete categories and $G : \mathcal{B} \rightarrow \mathcal{C}$ is continuous - i.e. preserves small limits - then $(F \downarrow G)$ is finitely complete and both forgetful functors are continuous.

Proof. Let's begin by proving the cocomplete case. We will start from co-completeness of the category and then we will move on to the cocontinuity of the forgetful functors. Recall that a category is finitely cocomplete if and only if it admits initial object and pushouts; we will construct explicitly both objects:

Initial Object: We will construct the initial object of $(F \downarrow G)$ by using the cocompleteness of \mathcal{A} and \mathcal{B} : indeed, let $0_{\mathcal{A}}$ be the initial object of \mathcal{A} and $0_{\mathcal{B}}$ the initial object of \mathcal{B} . Now, since F is a cocontinuous functor $F(0_{\mathcal{A}}) = 0_{\mathcal{C}}$ where $0_{\mathcal{C}}$ is in \mathcal{C} ; thus there is a unique morphism $in : F(0_{\mathcal{A}}) = 0_{\mathcal{C}} \rightarrow G(0_{\mathcal{B}})$ in \mathcal{C} . Now, we claim $(0_{\mathcal{A}}, 0_{\mathcal{B}}, in)$ to be initial in $(F \downarrow G)$:

- existence: let (A, B, f) be an object of $(F \downarrow G)$; then there exists two unique morphisms in_A, in_B respectively in \mathcal{A} and \mathcal{B} such that $in_A : 0_{\mathcal{A}} \rightarrow A$ and $in_B : 0_{\mathcal{B}} \rightarrow B$. Thus we have a candidate morphism (in_A, in_B) . In order to check that this morphism is a morphism of $(F \downarrow G)$ we need to show that the following diagram is commutative:

$$\begin{array}{ccc} F(0_{\mathcal{A}}) = 0_{\mathcal{C}} & \xrightarrow{\quad in \quad} & G(0_{\mathcal{B}}) \\ \downarrow Fin_A & & \downarrow Gin_B \\ FA & \xrightarrow{\quad f \quad} & GB \end{array}$$

The diagram above is trivially commutative: since $0_{\mathcal{C}}$ is initial in \mathcal{C} both $f \circ Fin_A : 0_{\mathcal{C}} \rightarrow GB$ and $Gin_B \circ in : 0_{\mathcal{C}} \rightarrow GB$ must be equal to $in_{GB} : 0_{\mathcal{C}} \rightarrow GB$. Thus

$$(0_{\mathcal{A}}, 0_{\mathcal{B}}, in) \xrightarrow{(in_A, in_B)} (A, B, f)$$

is a morphism in $(F \downarrow G)$

- uniqueness: let $(0_{\mathcal{A}}, 0_{\mathcal{B}}, in) \xrightarrow[(h,k)]{(in_A, in_B)} (A, B, f)$ two arrows in $(F \downarrow G)$; then since $0_{\mathcal{A}}, 0_{\mathcal{B}}$ are initial, then $h = in_A$ and $k = in_B$; hence $(h, k) = (in_A, in_B)$ and so (in_A, in_B) is unique

Thus $(F \downarrow G)$ admits initial object

Pushouts: As for the case above let construct the pushouts in $(F \downarrow G)$ by using pushout provided by cocompleteness of \mathcal{A}, \mathcal{B} . Take the following scenario in $(F \downarrow G)$

$$\begin{array}{ccc} (A, B, f) & \xrightarrow[(h,k)]{} & (A', B', f') \\ \downarrow (h', k') & & \\ (A'', B'', f'') & & \end{array} \quad (1)$$

then we can form two pushouts in \mathcal{A} and \mathcal{B} :

$$\begin{array}{ccc} A & \xrightarrow{h} & A' \\ \downarrow h' & & \downarrow p_1 \\ A'' & \xrightarrow{p_2} & P_A \end{array} \quad \begin{array}{ccc} B & \xrightarrow{k} & B' \\ \downarrow k' & & \downarrow q_1 \\ A'' & \xrightarrow{q_2} & P_B \end{array}$$

Now, following the same idea of initial object we may expect the pushout in $(F \downarrow G)$ to be (P_A, P_B, u) for some $u : F(P_A) \rightarrow G(P_B)$. Now, we need to construct a candidate for u . First of all, since F preserves finite colimits, $F(P_A) = P'_A$ is the pushout of $Fh : FA \rightarrow FA'$ and $Fh' : FA \rightarrow FA''$ in \mathcal{C} (with projection $\pi_1 : FA' \rightarrow P'_A, \pi_2 : FA'' \rightarrow P'_A$ equal to Fp_1 and Fp_2 up to

isomorphism). Then we have the following diagram:

$$\begin{array}{ccc}
 FA & \xrightarrow{Fh} & FA' \\
 \downarrow Fh' & \nearrow & \downarrow Fq_1 \circ f' \\
 & P'_A & \\
 & \searrow \exists! u & \\
 FA'' & \xrightarrow{Fq_2 \circ f''} & GP_B
 \end{array} \quad (2)$$

The external square is commutative since $Fq_1 \circ f' \circ Fh = Fq_1 \circ Fk \circ f = Fq_2 \circ Fk' \circ f = Fq_2 \circ f'' \circ Fh'$ (we used arrow condition of $(F \downarrow G)$ plus the fact that image under functor of commutative diagrams are indeed commutative). Thus there exists a unique $u : P'_A \rightarrow GP_B$ such $u \circ Fp_2 = Fq_2 \circ f''$ and $u \circ Fp_1 = Fp_1 \circ f'$. Now, we claim (P_A, P_B, u) to be the pushout of (1) with projections $(p_1, q_1) : (A', B', f') \rightarrow (P_A, P_B, u)$ and $(p_2, q_2) : (A'', B'', f'') \rightarrow (P_A, P_B, u)$.

$$\begin{array}{ccc}
 (A, B, f) & \xrightarrow{(h,k)} & (A', B', f') \\
 \downarrow (h',k') & & \downarrow (p_1,q_1) \\
 (A'', B'', f'') & \xrightarrow{(p_2,q_2)} & (P_A, P_B, u)
 \end{array}$$

Now, we shall prove that the projections are actually arrows of $(F \downarrow G)$ and that the universal property holds.

We begin proving that the projections are actually arrow, i.e. that the following diagrams are commutative:

$$\begin{array}{ccc}
 FA' & \xrightarrow{f'} & GB' \\
 \downarrow Fp_1 & & \downarrow Gq_1 \\
 P'_A & \xrightarrow{u} & GP_B
 \end{array}
 \quad
 \begin{array}{ccc}
 FA'' & \xrightarrow{f''} & GB'' \\
 \downarrow Fp_2 & & \downarrow Gq_2 \\
 P'_A & \xrightarrow{u} & GP_B
 \end{array}$$

Both diagrams are commutative as consequence of the diagram (2); hence projections are indeed arrows in $(F \downarrow G)$.

Now we prove that the universal property holds:

- existence: assume the following diagram is commutative

$$\begin{array}{ccc}
 (A, B, f) & \xrightarrow{(h,k)} & (A', B', f') \\
 \downarrow (h',k') & & \downarrow (x',y') \\
 (A'', B'', f'') & \xrightarrow{(x'',y'')} & (X, Y, g)
 \end{array}$$

First notice that since (x', y') and (x'', y'') are arrows of $(F \downarrow G)$ then

$$g \cdot Fx' = Gy' \cdot f' \quad (3)$$

$$g \cdot Fx'' = Gy'' \cdot f'' \quad (4)$$

Second, since the square is commutative, then we have the following two commutative diagram in \mathcal{A} and \mathcal{B} thanks to the universal property of P_A and P_B :

$$\begin{array}{ccc}
 A & \xrightarrow{h} & A' \\
 \downarrow h' & \nearrow p_1 & \downarrow x' \\
 & P_A & \\
 \nearrow p_2 & \searrow \exists! u_A & \\
 A'' & \xrightarrow{x''} & X
 \end{array}
 \quad
 \begin{array}{ccc}
 B & \xrightarrow{k} & B' \\
 \downarrow k' & \nearrow q_1 & \downarrow y' \\
 & P_B & \\
 \nearrow q_2 & \searrow \exists! u_B & \\
 A'' & \xrightarrow{y''} & Y
 \end{array}
 \quad (5)$$

Now, we claim that the following diagram is commutative:

$$\begin{array}{ccc}
 (A, B, f) & \xrightarrow{\quad\quad\quad} & (A', B', f') \\
 \downarrow & \nearrow (p_1, q_1) & \downarrow (x', y') \\
 & (P_A, P_B, u) & \\
 \nearrow (p_2, q_2) & \searrow (u_a, u_b) & \\
 (A'', B'', f'') & \xrightarrow{(x'', y'')} & (X, Y, g)
 \end{array}
 \quad (6)$$

If (u_A, u_B) is an arrow of $(F \downarrow G)$ then we are done, since the square is commutative as a direct consequence of the commutativity of (5). Then we claim (u_A, u_B) to be an actual arrow of $(F \downarrow G)$, i.e. we claim

that the following diagram is commutative:

$$\begin{array}{ccc}
 P'_A & \xrightarrow{u} & GP_B \\
 Fu_A \downarrow & & \downarrow Gu_B \\
 FX & \xrightarrow{g} & GY
 \end{array} \tag{7}$$

We will prove commutativity using that P'_A is a pushout in \mathcal{C} . Indeed, consider the following diagram:

$$\begin{array}{ccccc}
 & & FA & & \\
 & \swarrow Fh' & & \searrow Fh'' & \\
 FA' & & & & FA'' \\
 & \searrow Fp_1 & & \swarrow Fp_2 & \\
 & & P'_A & \xrightarrow{u} & GP_B \\
 & & \downarrow Fu_A & \searrow \exists! v & \downarrow Gu_B \\
 FA' & \xrightarrow{Fx'} & FX & \xrightarrow{g} & GY \\
 & & & & \uparrow gFx'' \\
 & & & & FA''
 \end{array}$$

Then using (3) and (4) we can show that the external square is commutative:

$$\begin{aligned}
 gFx''Fh'' &= Gy''f''Fh' = Gy''Gk'f = Gy'Gkf \\
 gFx'Fh &= Gy'f'Fh = Gy'Gkf
 \end{aligned}$$

Thus, by P'_A being pushout in \mathcal{C} there exists a unique $v : P'_A \rightarrow GY$ such that $vFp_1 = gFx'$ and $vFp_2 = gFx''$. Now, consider $gFu_A, Gu_Bu : P'_A \rightarrow GY$, notice that:

$$\begin{aligned}
 gFu_AFp_1 &= gFx' \\
 gFu_BFp_2 &= gFx'' \\
 Gu_BuFp_1 &= Gu_BGq_1f' = Gy'f' = gFx' \\
 Gu_BuFp_2 &= Gu_BGq_2f'' = Gy''f'' = gFx''
 \end{aligned}$$

So by uniqueness of v we get $gFu_A = Gu_Bu$ and so, since the square (7) is commutative, then (u_A, u_B) is an arrow of $(F \downarrow G)$ and the diagram (6) is commutative.

- uniqueness: the uniqueness of (u_A, u_B) comes almost for free: indee if there was antoher arrow (l, m) satisfyng commutativity in (6), then, by universal property of pushouts $l = u_A$ in \mathcal{A} and $m = u_B$ in \mathcal{B} . Therefore we would have $(l, m) = (u_A, u_B)$

Thus (P_A, P_B, u) is the desired pushout and $(F \downarrow G)$ having initial object and pushouts is indeed finitely cocomplete.

The forgetfoul functors are cocontinous: the cocontinuity of Π_1, Π_2 arises immediatly from the structure of intial object and pushouts of $(F \downarrow G)$. Indeed take the intial object of $(F \downarrow G)$, $(0_A, 0_B, in)$; then $\Pi_1((0_A, 0_B, in)) = 0_A$ and $\Pi_2((0_A, 0_B, in)) = 0_B$ are respectively the initial object of \mathcal{A} and \mathcal{B} . Since the same happens for the pushouts of $(F \downarrow G)$ then both Π_1 and Π_2 preserves finite colimits.

Now, let's move on to the finitely complete case; so suppose that both \mathcal{A} and \mathcal{B} are finitely complete and that G is finitely continous. The structure of the proof is completely analogue ² and so we will omit some details.

Terminal Objects: following the structure of the cocomplete proof we claim $(1_A, 1_B, un)$, where $un : F(1_A) \rightarrow 1_C = G(1_B)$ is the unique morphism granted by $G(1_B)$ be terminal in \mathcal{C} , to be the terminal object in $(F \downarrow G)$

- existence: let (A, B, f) be an object of $(F \downarrow G)$; then there exists two unique morphisms un_A, un_B respectively in \mathcal{A} and \mathcal{B} such that $un_A : A \rightarrow 1_A$ and $un_B : B \rightarrow 1_B$. Thus we have a candidate morphism (un_A, un_B) . In order to check that this morphism is a morphism of $(F \downarrow G)$ we need to show that the following diagram is commutative:

$$\begin{array}{ccc} F1_A & \xrightarrow{un} & G1_B = 1_C \\ \downarrow F un_A & & \downarrow G un_B \\ FA & \xrightarrow{f} & GB \end{array}$$

The diagram above is trivially commutative since $G1_B = 1_C$ is terminal in \mathcal{C}

- uniqueness: the uniqueness is directly inherited by uniness of un_A and un_B in \mathcal{A} and \mathcal{B}

²Notice that we can also argue that, beeing $(F \downarrow G)^{op} \cong (F^{op} \downarrow G^{op})$, the complete case comes for free from the cocomplete case

Pullbacks: As for the pushouts we claim the pullback of the following diagram:

$$\begin{array}{ccc}
 & (A, B, f) & \\
 & \downarrow (h, k) & \\
 (A', B', f) & \xrightarrow{(h', k')} & (A'', B'', f'')
 \end{array}$$

to be (P_A, P_B, u) with projections $(p_1, q_1) : (P_A, P_B, u) \rightarrow (A, B, f)$ and $(p_2, q_2) : (P_A, P_B, u) \rightarrow (A', B', f')$ where (P_A, p_1, p_2) and (P_B, q_1, q_2) are pullbacks in \mathcal{A} and \mathcal{B} (Remember that $GP_B = P'_B$ is a pullback in \mathcal{C} as was for FP_A in the cocomplete case). $u : FP_A \rightarrow P'_B$ arises from the universal property of P'_B as shown in the following diagram (whose commutativity is ensured by arrow condition in $(F \downarrow G)$ plus pullbacks condition of (P_A, q_1, q_2) in \mathcal{A}):

$$\begin{array}{ccccc}
 P_A & & & & \\
 \searrow \exists! u & \xrightarrow{fFp_1} & & & \\
 & P'_B & \xrightarrow{Gq_1} & GB & \\
 \downarrow f'Fp_2 & \downarrow Gq_2 & & \downarrow Gk & \\
 & GB' & \xrightarrow{Gk'} & GB'' &
 \end{array}$$

From the commutativity of this diagram we deduce that (q_i, p_i) are actual arrows of $(F \downarrow G)$, indeed:

$$Gq_2u = f'Fp_2 \tag{8}$$

$$Gq_1u = fFp_1 \tag{9}$$

Now, we check that the universal property holds

- Existence: assume the external square to be commutative:

$$\begin{array}{ccccc}
 (X, Y, g) & & & & \\
 \searrow (x, y) & & & & \\
 & (P_A, P_B, u) & \xrightarrow{(p_1, q_1)} & (A, B, f) & \\
 \downarrow (x', y') & \downarrow (p_2, q_2) & & \downarrow (h, k) & \\
 & (A', B', f') & \xrightarrow{(h', k')} & (A'', B'', f'') &
 \end{array} \quad (10)$$

Then we can use universal property of P_A and P_B in \mathcal{A} and \mathcal{B} :

$$\begin{array}{ccc}
 X & \xrightarrow{x} & A \\
 \searrow \exists! u_A & \searrow & \downarrow h \\
 P_A & \xrightarrow{\quad} & A \\
 \downarrow & & \downarrow \\
 A' & \xrightarrow{h'} & A''
 \end{array}
 \qquad
 \begin{array}{ccc}
 Y & \xrightarrow{y} & B \\
 \searrow \exists! u_B & \searrow & \downarrow k \\
 P_B & \xrightarrow{\quad} & B \\
 \downarrow & & \downarrow \\
 B' & \xrightarrow{k'} & B''
 \end{array} \quad (11)$$

Now, consider the couple (u_A, u_B) ; by commutativity of (11) if $(u_A, u_B) : (X, Y, g) \rightarrow (P_A, P_B, u)$ is an actual arrow of $(F \downarrow G)$ then it complete the diagram (10) to a commutative diagram.

We procede as in the case of pushouts; consider the following square:

$$\begin{array}{ccccc}
 FX & \xrightarrow{g} & GY & & \\
 \downarrow & \searrow \exists! v & \downarrow & \searrow Gg & \\
 FP_A & \xrightarrow{u} & Gu_B \cong P'_B & \xrightarrow{Gq_1} & GB \\
 \downarrow Fx' & & \downarrow Gq_2 & & \downarrow Gk \\
 FA' & \xrightarrow{f'} & GB' & \xrightarrow{Gk'} & GB''
 \end{array}$$

Since (x', y') and (x, y) are arrows of $(F \downarrow G)$, the square (10) is commutative and using the definition of (p_i, q_i) we can show that the external square commutes, indeed:

$$Gk'f'Fx' = Gk'Gy'g = GkGyg \quad (12)$$

Thus, since P'_B is the pullback of $GB \xrightarrow{Gk} GB''$ and $GB' \xrightarrow{Gk'} GB''$ there exists an unique $v : FX \rightarrow P'_B$ such $Gq_1v = Gyg$ and $Gq_2v = f'Fx'$. Now, consider $uFu_A, Gu_Bg : FX \rightarrow P'_B$ and notice that:

$$\begin{aligned} Gq_1uFu_A &= fFp_1Fu_A = fFx = Gyg \\ Gq_2uFu_A &= fFp_2Fu_A = f'Fp_2Fu_A = f'Fx' \\ Gq_1Gu_Bg &= Gyg \\ Gq_2Gu_Bg &= Gy'g = f'Fx' \end{aligned}$$

Thus by uniqueness of $v : FX \rightarrow P'_B$ then $uFu_A = Gu_Bg$ and so $(u_A, u_B) : (X, Y, g) \rightarrow (P_A, P_B, u)$ is an arrow of $(F \downarrow G)$ and complete the diagram (10) to a commutative diagram

- uniqueness: uniqueness of u_A and u_B in \mathcal{A} and \mathcal{B} yields to uniqueness of (u_A, u_B) in $(F \downarrow G)$

Forgetful functors are continous: the proof is analogue to the cocomplete case, since both Π_1, Π_2 preserves terminal object and pullbacks.

We proved that under the assumption of \mathcal{A} and \mathcal{B} beeing cocomplete and F beeing cocontinous the comma category $(F \downarrow G)$ is cocomplete and that Π_i is cocontinous; morevoer we proved the dual result with complteness assumption, therefore the statement is proved and we are done. \square

Definition 0.2.4 (Freyd Cover). Let \mathcal{C} be a small category (serve veramente sia piccola?) with a terminal object 1, then we call its *Freyd Cover*, $Cov(\mathcal{C})$, the comma category $Id \downarrow \mathcal{C}(1, -)$ where $Id : Set \rightarrow Set$ is the identical functor

Definition 0.2.5 (Projection Functor of Freyd Cover). Recall that there are two canonical functors Π_1, Π_2 associated with a comma category. Let \mathcal{C} be a category with terminal object and let $Cov(\mathcal{C})$ be its Frey Cover. We call the functor $\Pi_2 : Cov(\mathcal{C}) \rightarrow \mathcal{C}$ the projection functor of the Freyd Cover.

Theorem 0.2.6. Let \mathcal{C} be a bicartesian category - i.e. a category with terminal object, initial object and binary products and coproducs - then $Cov(\mathcal{C})$ is bicartesian. Moreover $\Pi_2 : Cov(\mathcal{C}) \rightarrow \mathcal{C}$ is a cartesian closed functor

Proof. The proof follows a similar structure to the proof of 0.2.3, so we will construct initial and terminal objects, binary products and copruducts and then we will prove that Π_2 is closed with respect to this constructions.

Initial and Terminal Object: the initial object of $Cov(\mathcal{C})$ is $(0_{Set}, 0_{\mathcal{C}}, in)$

and the terminal object is $(1_{Set}, 1_{\mathcal{C}}, un)$. The construction is the same of the construction in 0.2.3 and therefore the proof is omitted

Binary Products: Recall that $\mathcal{C}(1_{\mathcal{C}}, -) : \mathcal{C} \rightarrow Set$ is a continuous functor - i.e. preserves limits - and so in particular it preserves binary products. Therefore if A, B are objects of \mathcal{C} we have a set isomorphism σ :

$$\mathcal{C}(1_{\mathcal{C}}, A) \times \mathcal{C}(1_{\mathcal{C}}, B) \longrightarrow \mathcal{C}(1_{\mathcal{C}}, A \times B)$$

$$(h, k) \longmapsto \langle h, k \rangle : 1_{\mathcal{C}} \rightarrow A \times B$$

Now, let $(X, A, f), (Y, B, g)$ be two object of $Cov(\mathcal{C})$; we claim their product to be $(X \times Y, A \times B, \sigma \circ (f \times g))$ where $\sigma \circ (f \times f)$ arises from the following product diagram in Set :

$$\begin{array}{ccccc} & & X & \xrightarrow{f} & \mathcal{C}(1_{\mathcal{C}}, A) \\ & \nearrow \pi_X & & & \nwarrow \pi_1 \\ X \times Y & \xrightarrow{\exists! \langle f\pi_x, g\pi_y \rangle := f \times g} & \mathcal{C}(1_{\mathcal{C}}, A) \times \mathcal{C}(1_{\mathcal{C}}, A) & \xrightarrow{\sigma} & \mathcal{C}(1_{\mathcal{C}}, A \times B) \\ & \searrow \pi_Y & & & \swarrow \pi_2 \\ & & Y & \xrightarrow{g} & \mathcal{C}(1_{\mathcal{C}}, B) \end{array}$$

and we claim the projections of $(X \times Y, A \times B, \sigma \circ (f \times g))$ to be $(\pi_X, \pi_A) : (X \times Y, A \times B, \sigma \circ (f \times g)) \rightarrow (X, A, f)$ and $(\pi_Y, \pi_B) : (X \times Y, A \times B, \sigma \circ (f \times g)) \rightarrow (Y, B, g)$.

We start by checking that our candidate projections are arrows of $Cov(\mathcal{C})$ and then we will check that the universal property holds. In order to check that our projection are actual arrows we shall prove that $\mathcal{C}(1_{\mathcal{C}}, \pi_A)\sigma(f \times g) = f\pi_X$ and that $\mathcal{C}(1_{\mathcal{C}}, \pi_B)\sigma(f \times g) = \pi_B g$ but this two identities follow directly from the fact that $\mathcal{C}(1_{\mathcal{C}}, -)$ preserves products and that products are unique up to isomorphism, thus $\pi_1 = \mathcal{C}(\mathcal{C}, -)\pi_A$ and $\pi_2 = \mathcal{C}(\mathcal{C}, -)\pi_B$ and so the identity are trivial. Let's move on to the universal property.

- existence: let consider the following situation:

$$\begin{array}{ccccc} & & (X \times Y, A \times B, \sigma(f \times g)) & & \\ & \swarrow (\pi_X, \pi_A) & & \searrow (\pi_Y, \pi_B) & \\ (X, A, f) & \xleftarrow{(l, m)} & (Z, C, h) & \xrightarrow{(l', m')} & (Y, B, g) \end{array} \quad (13)$$

then, since $X \times Y$ and $A \times B$ are product respectively in Set and \mathcal{C} we have the following commutative diagrams:

$$\begin{array}{ccc} & X \times Y & \\ \pi_X \swarrow & \uparrow \exists! \bar{l} & \searrow \pi_Y \\ X & \xleftarrow{l} Z \xrightarrow{l'} & Y \end{array} \quad \begin{array}{ccc} & A \times B & \\ \pi_A \swarrow & \uparrow \exists! \bar{m} & \searrow \pi_B \\ A & \xleftarrow{m} C \xrightarrow{m'} & B \end{array}$$

Thus consider the couple (\bar{l}, \bar{m}) ; if it is a morphism of $Cov(\mathcal{C})$ then it completes the diagram (13) to a commutative diagram. Now, consider the following diagram:

$$\begin{array}{ccccc} & Z & \xrightarrow{h} & \mathcal{C}(1_{\mathcal{C}}, C) & \\ & \downarrow \bar{l} & & \downarrow \mathcal{C}(1_{\mathcal{C}}, \bar{m}) & \\ Y & \xleftarrow{l'} X \times Y & \xrightarrow{\quad} & \mathcal{C}(1_{\mathcal{C}}, A \times B) & \\ & \searrow f \times g & \searrow \sigma \left(\begin{smallmatrix} \cong \\ \downarrow \end{smallmatrix} \right) \sigma^{-1} & & \\ & \mathcal{C}(1_{\mathcal{C}}, B) & \xleftarrow{\pi_2} \mathcal{C}(1_{\mathcal{C}}, A) \times \mathcal{C}(1_{\mathcal{C}}, B) & \xrightarrow{\pi_1} \mathcal{C}(1_{\mathcal{C}}, A) & \end{array}$$

Since $\pi_1 \sigma^{-1} \mathcal{C}(1_{\mathcal{C}}, \bar{m}) h = \mathcal{C}(1_{\mathcal{C}}, m) h = \pi_1 (f \times g) \bar{l}$ and $\pi_2 \sigma^{-1} \mathcal{C}(1_{\mathcal{C}}, \bar{m}) h = g l' = \pi_2 (f \times g) \bar{l}$ then by the universal property of binary product $\sigma^{-1} \mathcal{C}(1_{\mathcal{C}}, \bar{m}) h = \sigma^{-1} \sigma (f \times g) \bar{l}$ hence $\sigma (f \times g) \bar{l} = \mathcal{C}(1_{\mathcal{C}}, \bar{m}) h$ thus (\bar{l}, \bar{m}) is an arrow of $Cov(\mathcal{C})$ and complete the diagram (13) to a commutative diagram

- Uniqueness: uniqueness of (\bar{l}, \bar{m}) is directly inherited by uniqueness of \bar{l} and \bar{m} in \mathcal{A} and \mathcal{B}

Binary Coproducts: Let (X, A, f) and (Y, B, g) be two objects of $Cov(\mathcal{C})$, then we claim their coprduct to be $(X \oplus Y, A \oplus B, f \oplus g)$ with injections (ι_X, ι_A) and (ι_Y, ι_B) where $f \oplus g$ arises from:

$$\begin{array}{ccc} X & \xrightarrow{f} & \mathcal{C}(1_{\mathcal{C}}, A) \\ \iota_X \swarrow & & \searrow (\iota_1)_{\mathcal{C}} \\ X \oplus Y & \xrightarrow{\exists! f \oplus g} & \mathcal{C}(1_{\mathcal{C}}, A \oplus B) \\ \iota_Y \swarrow & & \searrow (\iota_2)_{\mathcal{C}} \\ Y & \xrightarrow{g} & \mathcal{C}(1_{\mathcal{C}}, B) \end{array}$$

The proof is dual to the proof of binary product and so we omit the calculations.

Projection is cartesian closed: the proof of Π_2 beeing cartesian closed functor is identical to the proof of Π_1 beeing continous in 0.2.3 \square

Remark 0.2.7. Notice that this theorem can be generalized for arbitrary comma categories with appropriate assumptions. In fact the construction process of products and coproduct of this proof can be generalized with the same approach we used in the proof of 0.2.3.

Theorem 0.2.8. Let \mathcal{H} be an Heyting Algebra - i.e. an ordered bicartesian category which is also cartesian closed - then $Cov(\mathcal{H})$ is also bicartesian closed. Moreover the projection $\Pi_2 : Cov(\mathcal{H}) \rightarrow \mathcal{H}$ is bicartesian closed.

Proof. By 0.2.6 we know that $Cov(\mathcal{H})$ is bicartesian and that Π_2 is cartesian closed. So we need only to prove the existence of function spaces.

Let (X, a, f) and (Y, b, g) be objects of $Cov(\mathcal{C})$. We expect the exponential $(Y, b, g)^{(X, a, f)}$ to be something similar to $(Y^X, a \rightarrow b, \xi)$ where $Y^X = Set(X, Y)$ is the function space in Set , $a \rightarrow b$ is the heyting implication in \mathcal{H} and ξ is a function somehow defined using that beeing \mathcal{H} an ordered category if $a \rightarrow b$ in nonempty the in terminal in Set . This intuition is almost correct, but there is a problem: in order to make $(Y^X, a \rightarrow b, \xi)$ an object of \mathcal{H} we need $\xi : Y^X \rightarrow \mathcal{H}(1_{\mathcal{H}}, a \rightarrow b)$ to be a Set -morphism which is not possible if $1_{\mathcal{H}} \not\leq_{\mathcal{H}} a \rightarrow b$ and $Y^X \neq \emptyset$. Thus we claim $(Y, b, g)^{(X, a, f)}$ to be $((Y^X)_{a \leq b}, a \rightarrow b, \xi)$ where $(Y^X)_{a \leq b}$ is $Set(X, Y)$ if $a \leq_{\mathcal{H}} b$ - and so $\mathcal{H}(1_{\mathcal{H}}, a \rightarrow b) \neq \emptyset$ - and to be empty otherwise:

$$(Y^X)_{a \leq b} := \{h \in Y^X \mid \forall x \in X (f(x) \in \mathcal{H}(1_{\mathcal{H}}, a) \implies g(h(x)) \in \mathcal{H}(1_{\mathcal{H}}, b))\}$$

with evaluation given by: $(Ev_{Set}, Ev_{\mathcal{H}}) : ((Y^X)_{a \leq b}, a \rightarrow b, \xi) \rightarrow (Y, b, g)$ where $Ev_{Set} : (Y^X)_{a \leq b} \times X \rightarrow Y$ is the restriction of usual evaluation morphism in Set and $Ev_{\mathcal{H}} = a \rightarrow b \wedge a \leq b$ is the counit of the adjunction of exponential functor in \mathcal{H} . Now, let $(u, c \wedge a \leq b) : (Z, c, h) \times (X, a, f) \rightarrow (Y, b, g)$, we claim its abstraction to be:

$$(\hat{u}, c \leq a \rightarrow b) : (Z, c, h) \rightarrow ((Y^X)_{a \leq b}, a \rightarrow b, \xi)$$

This couple is clearly an arrow of $Cov(\mathcal{C})$ ³. We claim that, if the abstraction

³the arrow condition is forced since if every definition is coherent than the commutativity of the arrow condition square follows from $\mathcal{H}(1_{\mathcal{H}}, a \rightarrow b)$ be either termial or empty

is well defined, then the following diagram is commutative:

$$\begin{array}{ccc}
 (X, a, f) \times (Z, c, h) & & \\
 \downarrow (u, c \leq a \rightarrow b) & \searrow (u, c \wedge a \leq b) & \\
 ((Y^X)_{a \leq b}, a \rightarrow b, \xi) & \xrightarrow{(Ev_{Set}, Ev_{\mathcal{H}})} & (Y)
 \end{array} \tag{14}$$

Now, commutativity at the level of *Set* is ensured by universal property of Y^X in *Set* plus the coeherence condition of $(Y^X)_{a \leq b}$ with respect to $a \rightarrow b$. Now, in \mathcal{H} we have that $c \wedge a \leq b$ we have that $c \wedge a \leq b \iff c \leq a \rightarrow b$ so we have commutativity at the level of \mathcal{H} . It remains to check the well definition of the abstraction: let $Z \neq \emptyset$, then $h(m) \in \mathcal{H}(1_{\mathcal{H}}, c)$ for $m \in Z$, hence $1_{\mathcal{H}} \leq c$ and so, since $c \wedge a \leq b$, then $a \leq b$ and so $\hat{u}(m) \in (Y^X)_{a \leq b}$ which ensure well definition of abstraction arrows. Notice that uniqueness of abstraction follows directly from uniqueness of abstractions in *Set* and \mathcal{H} . Thus $Cov(\mathcal{H})$ is a bicartesian closed category.

Concerning Π_2 , the proof is identical to 0.2.3 since the \mathcal{H} level of the exponential object is the exponential object in \mathcal{H} and so Π_2 is a bicartesian closed functor and we are done. \square

Definition 0.2.9. Let \mathcal{H} be an Heyting Algebra, then we call its Freyd cover the Frey-Heyting cover of \mathcal{H}

Definition 0.2.10. We say that **biCC** is the category with objects bicartesian closed categories and morphisms functors preserving the bicartesian structure up to isomorphism

Definition 0.2.11 (Global Set Functor). Let \mathcal{C} be a locally small category with terminal object $1_{\mathcal{C}}$. We define the global set functor as:

$$\begin{array}{ccc}
 \mathcal{C} & \xrightarrow{\Gamma(-)} & Set \\
 \\
 A & \xrightarrow{\Gamma(-)} & \Gamma(A) = \mathcal{C}(1_{\mathcal{C}}, A) \\
 \downarrow & \xrightarrow{\quad} & \downarrow \Gamma(f) = \mathcal{C}(1_{\mathcal{C}}, f) \\
 B & & \Gamma(B) = \mathcal{C}(1_{\mathcal{C}}, B)
 \end{array}$$

Lemma 0.2.12. Let \mathcal{C} be a category in **biCC**, then its freyd cover $Cov(\mathcal{C})$ is again in **biCC**

Proof. We want to generalize construction in 0.2 for arbitrary bicartesian closed category. We mimic the construction in [1]. Let \mathcal{C} be a bicartesian closed category and let (X, A, f) and (Y, B, g) , we want to construct the exponential object $(Y, B, g)^{(X, A, f)}$. As in 0.2 we expect the exponential to mimic (Y^X, B^C, h) where $h : Y^X \rightarrow \Gamma(B^C)$; however we know that (Y^X) cannot be the right *Set*-component of the exponential. moreover the constuction of h in this situation is slithely more complex. Let's begin by noticing that the global set functor is a product preserving functor and that, since \mathcal{C} is CC, in \mathcal{C} there is the exponential object B^A with evaluation $ev_{\mathcal{C}} : B^A \times B \rightarrow A$. We can form the composition :

$$u : \Gamma(A) \times \Gamma(B^A) \xrightarrow{\cong} \Gamma(A \times B^A) \xrightarrow{\Gamma(ev_{\mathcal{C}})} \Gamma(B) \quad (15)$$

Now, we take the abstraction of u in *Set*, $\hat{u} : \Gamma(A) \rightarrow \Gamma(B)^{\Gamma(A)}$. Consider now the following pullback diagram:

$$\begin{array}{ccc} H & \xrightarrow{p_2} & Y^X \\ \downarrow p_1 & \lrcorner & \downarrow g^X \\ \Gamma(B^A) & \xrightarrow{\quad} & \Gamma(B)^X \\ & \searrow \hat{u} & \nearrow (\Gamma(B))^f \\ & \Gamma(B)^{\Gamma(A)} & \end{array}$$

Where $(\Gamma(D))^f : \Gamma(B)^{\Gamma(A)} \rightarrow \Gamma(D)^X$ is defined by $\mu \rightarrow \mu \circ f$ and $g^X : Y^X \rightarrow \Gamma(B)$ is defined by $\omega \rightarrow g \circ \omega$. We claim $(H, B^A, h := p_1)$ to be the exponenetial object $(X, A, f)^{(Y, B, g)}$. We will prove this by checking that, provided $E = (Z, E, k)$, we have a natural ismoporphism $Cov(\mathcal{C})(E \times A, B) \cong Cov(\mathcal{C})(E, B^A)^4$.

Construction of $\Phi : Cov(\mathcal{C})(E \times A, B) \rightarrow Cov(\mathcal{C})(E, B^A)$: let $(\psi_{Set}, \psi_{\mathcal{C}})$ in $Cov(\mathcal{C})(E \times A, B)$ then, $\psi_{Set} : Z \times X \rightarrow Y$ and $\psi_{\mathcal{C}} : E \times A \rightarrow B$. Thus, using that *Set* and \mathcal{C} are cartesian closed, we can take the abstractions

⁴here we make an abuse of notation setting $A := (X, A, f), B := (Y, B, g)$

$\hat{\psi}_{Set} : Z \rightarrow Y^X$ and $\hat{\psi}_C : E \rightarrow B^A$. Now, consider the following diagram:

$$\begin{array}{ccccc}
 Z & & & & \\
 \downarrow k & \searrow \exists! \phi_{Set} & & \searrow \hat{\psi}_{Set} & \\
 & H & \xrightarrow{p_2} & Y^X & \\
 & \downarrow p_1 & \lrcorner & \downarrow g^X & \\
 \Gamma(E) & & & & \\
 \searrow \Gamma(\hat{\psi}_C) & & & & \\
 & \Gamma(B^A) & \xrightarrow{\Gamma(D)^f \circ \hat{u}} & \Gamma(D)^X &
 \end{array}$$

we can check that the external square commutes using unicity of abstraction; so we check that:

$$Ev_{Set} \circ (g^X \circ \hat{\psi}_{Set} \times id_X) = Ev_{Set} \circ ((\Gamma B)^f \circ \hat{u} \circ \Gamma(\hat{\psi}_C) \circ k \times id_X) \quad (16)$$

Indeed:

$$Ev_{Set} \circ (g^X \circ \hat{\psi}_{Set} \times id_X) = Ev_{Set} \circ (g^X \times id_X) \circ (\hat{\psi}_{Set} \times id_X) = g \psi_{Set}$$

The last equality can be seen using explicit definition of g^X ; we have that any $l : z \rightarrow Y^X$, $g(l) : z \rightarrow g(l(z)) \in \Gamma(B)^X$ and so $Ev_{Set}(g^X \times id_X)$ sends (l, x) to $g(l(x))$. Also $g \circ Ev_{Set}$ sends (l, x) to $g(l(x))$ and thus $Ev_{Set} \circ (g^X \times id_X) = g \circ Ev_{Set}$ and so the equality is justified. Let's move on to the right side of the equation:

Let $\psi_2 := \Gamma(\hat{\psi}_C)$ and $t := \hat{u} \circ \psi : Z \rightarrow \Gamma B^{\Gamma A}$. Then the function $\Gamma B^f \circ t$ assign to each $z \in Z$ the function $t(z) \circ f : X \rightarrow \Gamma B$. Thus the evaluation is computed as follows:

$$Ev_{Set}(\Gamma B^f t(z), x) = (t(z) \circ f)(x) = (t(z)f)(x)$$

this yields to:

$$Ev_{Set}((\Gamma B^f \circ t) \times id_X) = Ev_{Set} \circ (t \times f) : Z \times X \rightarrow \Gamma B$$

Thus substituting t we obtain:

$$\begin{aligned}
 Ev_{Set} \circ ((\Gamma B)^f \circ \hat{u} \circ \Gamma(\hat{\psi}_C) \circ k \times id_X) &= Ev_{Set} \circ ((\hat{u} \circ \psi_2) \times f) = \\
 &= Ev_{Set} \circ (\hat{u} \times id_{\Gamma A}) \circ (\psi_2 \times f)
 \end{aligned}$$

Now, recalling that \hat{u} is the abstraction of $\Gamma(Ev_C) \circ \theta : \Gamma A \times \Gamma(B^A) \rightarrow \Gamma B$ where $\theta : \Gamma A \times \Gamma(B^A) \rightarrow \Gamma(A \times B^A)$ is the canonical isomorphism provided by the fact that the global set functor preserves products we have:

$$Ev_{Set} \circ (\hat{u} \times id_{\Gamma A}) = \Gamma Ev_{Set} \circ \theta$$

Therefore:

$$Ev_{Set} \circ (\hat{u} \times id_{\Gamma A}) \circ (\psi_2 \times f) = \Gamma(Ev_{Set}) \circ \theta \circ (\psi_2 \times f)$$

Now, expliciting the definition of θ we get :

$$Ev_{Set} \circ ((\Gamma B^f \circ \hat{u} \circ \psi_2) \times id_X) = \Gamma(\psi_C) \circ (k \times f)$$

Now, the arrow condition on (ψ_{Set}, ψ_C) proves the equation 16, therefore by the universal property of H there exists a unique $\phi_{Set} : Z \rightarrow H$ that makes the pullback diagram commutative. Notice that the universal property of pullbacks ensure us that $(\phi_{Set}, \hat{\psi}_C)$ is an element of $Cov(\mathcal{C})(E, B^A)$. Moreover, this construction is unique, hence we have defined a function $\Phi_1 : Cov(\mathcal{C})(E \times A, B) \rightarrow Cov(\mathcal{C})(E, B^A)$ sending (ψ_{Set}, ψ_C) to $(\phi_{Set}, \hat{\psi}_C)$.

Contrution of $\Psi : \mathbf{Cov}(\mathcal{C})(\mathbf{E}, \mathbf{B}^A) \rightarrow \mathbf{Cov}(\mathcal{C})(\mathbf{E} \times \mathbf{A}, \mathbf{B})$: let (ψ_{Set}, ψ_C) in $Cov(\mathcal{C})(E, B^A)$, then the idea is to performe and "inverse construction" with respect to Φ . Let's begin with the *Set*-component.

We have $\psi_{Set} : Z \rightarrow H$, thus we define $\phi_1 := p_1 \circ \psi_{Set}$ and $\phi_2 := p_2 \circ \psi_{Set}$. Now, we use the exponential structure of *Set* to define

$$\bar{\psi}_{Set} := Ev_{Set} \circ (\phi_1 \times id_X) : Z \times X \rightarrow Y$$

Now, we mimic this construction in \mathcal{C} setting

$$\bar{\psi}_C := Ev_C \circ (\psi_C \times id_A) : E \times A \rightarrow B$$

We claim that $(\bar{\psi}_{Set}, \bar{\psi}_C)$ is an arrow of $Cov(\mathcal{C})$ i.e that

$$g\bar{\psi}_{Set} = \Gamma\bar{\psi}_C(k \times f) : Z \times X \rightarrow \Gamma B \quad (17)$$

Now, we begin unfolding the left side of the equation⁵:

$$\begin{aligned} g\bar{\psi}_{Set} &= g \circ Ev_{Set} \circ (\phi_1 \times id_X) = Ev_{Set} \circ (g^X \times id_X) \circ (\phi_1 \times id_X) \\ &= Ev_{Set} \circ ((g^X \circ \phi_1) \times id_X) \end{aligned}$$

Now, by definition ϕ_1 :

$$g^X \circ \phi_1 = (\Gamma B)^f \circ \hat{u} \circ \psi_2$$

hence:

$$g\bar{\psi}_{Set} = Ev_{Set} \circ ((\Gamma B^f \circ \hat{u} \circ \Gamma\psi_C \circ k) \times id_X)$$

⁵we use some identities derived in the last step

Now, with same argument used in 16

$$Ev_{Set} \circ ((\Gamma B^f \circ \hat{u} \circ \Gamma \psi_{\mathcal{C}} \circ k) \times id_X) = Ev_{Set} \circ ((\hat{u} \circ \Gamma \psi_{\mathcal{C}} \circ k) \times f)$$

hence substituting:

$$g\bar{\psi}_{Set} = Ev_{Set} \circ (u \times id_{\Gamma A}) \circ (\Gamma \psi_{\mathcal{C}} \circ k, f)$$

Now, recalling the definition of \hat{u} and expliciting the canonical isomorphism θ we get:

$$\begin{aligned} g \circ \bar{\psi}_{Set} &= \Gamma(Ev_{Set}) \circ \theta \circ (\Gamma \psi_{\mathcal{C}} k, f) \\ &= \Gamma \bar{\psi}_{\mathcal{C}} \circ (k, f) \end{aligned}$$

Thus, since the universal property of exponentials and pullbacks grant us the unicity of $(\hat{\psi}_{Set}, \hat{\psi}_{\mathcal{C}})$ then we have succesfully constructed a morphism $\Psi : Cov(\mathcal{C})(E, B^A) \rightarrow Cov(\mathcal{C})(E \times A, B)$.

Φ and Ψ gives a natural isomorphism: this follows easily by construction. In fact evaluation and abstractions are inverse to each other and, since we are working in *Set* with the two "projections" of set component the universal property of pullbacks grants us that our functions are inverse each other. Moreover our construction uses only mapping derived from natural isomorphisms - e.g. natural isomorphism of exponential objects in *Set* and \mathcal{C} and natural isomorphism of pullbacks in *Set*. Therefore our assignment is natural.

We proved that there is a natural isomorphism:

$$Cov(\mathcal{C})(E \times A, B) \cong Cov(\mathcal{C})(E, B^A)$$

And so the, since the functor $(-)^A$ is right adjoint to $(-) \times A$, then our candidate (H, B^A, h) is indeed the exponential objet. Thus $Cov(\mathcal{C})$ is cartesian closed and we are done. (AGGIUNGERE CONTI) \square

Remark 0.2.13. Notice that the exponential construction in 0.2.12 agrees with exponential construction in 0.2 if \mathcal{C} is an ordered catgory

Proposition 0.2.14. The Freyd Cover of a bicartesian closed category extend to an endofunctor

$$\mathbf{biCC} \xrightarrow{Cov(-)} \mathbf{biCC}$$

$$\begin{array}{ccc} \mathcal{A} & & Cov(\mathcal{A}) \\ F \downarrow & \xrightarrow{Cov(-)} & \downarrow Cov(F) \\ \mathcal{B} & & Cov(\mathcal{B}) \end{array}$$

where:

$$\begin{array}{ccc}
Cov(\mathcal{A}) & \xrightarrow{Cov(F)} & Cov(\mathcal{B}) \\
\\
(X, A, f) & \longmapsto & (X, FA, F \cdot f) \\
(h,k) \downarrow & \longmapsto & \downarrow (h, Fk) \\
(Y, B, g) & & (Y, FB, F \cdot g)
\end{array}$$

where $F \cdot f : X \rightarrow \Gamma(FA)$ is the set function defined sending $x \in X$ into the global setion of FA given by $F(f(x)) : 1_B \rightarrow FA$

Proof. By lemma 0.2.12 $Cov(-)$ exentend to a class function from **biCC** to itself. Then notice that the action of $Cov(-)$ over arrows of **biCC** is well defined since the commutativity condition of the arrow is preserved under the action of F . Indeed

$$Ff :$$

□

Bibliography

- [1] Carlo Angiuli. “Canonicity for simple types via gluing”. In: (2022).